CCGPS Frameworks Student Edition

Mathematics

CCGPS Analytic Geometry
Unit 4: Extending the Number System

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"Making Education Work for All Georgians"

Unit 4
Extending the Number System

Table of Contents

OVERVIEW

In this unit students will:

- Operate with polynomials with an emphasis on expressions that simplify to linear or quadratic forms.
- Define rational exponents
- Rewrite expression involving radicals and rational exponents
- Define the imaginary number *i*
- Define complex numbers
- Operate with complex numbers
- Understand that the basic properties of numbers continue to hold with polynomials and exponentials.

During the school-age years, students must repeatedly extend their conception of numbers. From counting numbers to fractions, students are continually updating their use and knowledge of numbers. In Grade 8, students extend this system once more by differentiating between rational and irrational numbers. In high school, students' number knowledge will be supplemented by the addition of the imaginary number system. The basic properties of numbers continue to hold as polynomial and exponential simplifications are explored.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

KEY STANDARDS

Extend the properties of exponents to rational exponents.

MCC9-12.N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

MCC9-12.N.RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

MCC9-12.N.RN.3 Explain why the sum or product of rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Perform arithmetic operations with complex numbers.

MCC9-12.N.CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.

MCC9-12.N.CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

MCC9-12.N.CN.3 (+) Find the conjugate of a complex number; use conjugates to find ~~moduli and~~ quotients of complex numbers.

Perform arithmetic operations on polynomials

MCC9-12.A.APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. *(Focus on polynomial expressions that simplify to forms that are linear or quadratic in a positive integer power of x.)*

RELATED STANDARDS

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain

insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

ENDURING UNDERSTANDINGS

- *Nth* roots are inverses of power functions. Understanding the properties of power functions and how inverses behave explains the properties of *nth* roots.
- Real-life situations are rarely modeled accurately using discrete data. It is often necessary to introduce rational exponents to model and make sense of a situation.
- Computing with rational exponents is no different from computing with integral exponents.
- The complex numbers are an extension of the real number system and have many useful applications.
- Addition and subtraction of complex numbers are similar to polynomial operations.

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- number sense
- computation with whole numbers and integers, including application of order of operations
- operations with algebraic expressions
- simplification of radicals
- measuring length and finding perimeter and area of rectangles and squares
- laws of exponents, especially the power rule

SELECTED TERMS AND SYMBOLS

According to Dr. Paul J. Riccomini, Associate Professor at Penn State University,

"*When vocabulary is not made a regular part of math class, we are indirectly saying it* ***isn't important!***" (Riccomini, 2008) Mathematical vocabulary can have significant positive and/or negative impact on students' mathematical performance.

- Require students to use mathematically correct terms.
- Teachers must use mathematically correct terms.
- Classroom tests must regularly include math vocabulary.
- Instructional time must be devoted to mathematical vocabulary.

http://www.nasd.k12.pa.us/pubs/SpecialED/PDEConference//Handout%20Riccomini%20Enhancing%20Math%20InstructionPP.pdf

The following terms and symbols are often misunderstood. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers. For help in teaching vocabulary, one technique which could be used is as follows:

Systematic Vocabulary Instruction; McREL 2008

Step 1: Present students with a brief explanation or description of the new term or phrase.

Step 2: Present students with a nonlinguistic representation of the new term or phrase.

Step 3: Ask students to generate their own explanation or description of the term or phrase.

Step 4: Ask students to create their own nonlinguistic representations of the term or phrase.

Step 5: Periodically ask students to review the accuracy of their explanations and representation.

http://www.mcrel.org/topics/products/340/

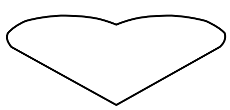
For example, the first definition covers the first two steps.

- **Complex number**: A complex number is the sum of a real number and an imaginary number (a number whose square is a real number less than zero), i.e. an expression of the form

  $a + bi$,
  where *a* and *b* are real numbers and *i* is the *imaginary unit*, satisfying $i^2 = -1$.

  Step 2:

  √-1

  MATH

- **Exponential functions:** A function of the form $y = a \cdot b^x$ where $a > 0$ and either $0 < b < 1$ or $b > 1$.

- **Expression:** A mathematical phrase involving at least one variable and sometimes numbers and operation symbols.

- ***Nth* roots:** The number that must be multiplied by itself *n* times to equal a given value. The *nth* root can be notated with radicals and indices or with rational exponents, i.e. $x^{1/3}$ means the cube root of *x*.

- **Polynomial function** A ***polynomial function*** is defined as a function, f(x)= $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-2}x^2 + a_{n-3}x^1 + a_n$, where the coefficients are real numbers.

- **Rational exponents:** For $a > 0$, and integers m and n, with $n > 0$, $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$; $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$.
- **Rational expression:** A quotient of two polynomials with a non-zero denominator.
- **Rational number**: A number expressible in the form a/b or $-\,a/b$ for some fraction a/b. The rational numbers include the integers.
- **Whole numbers**. The numbers 0, 1, 2, 3, ….

The properties of operations. Here a, b and c stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition $(a + b) + c = a + (b + c)$

Commutative property of addition $a + b = b + a$

Additive identity property of 0 $a + 0 = 0 + a = a$

Existence of additive inverses For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0$.

Associative property of multiplication $(a \times b) \times c = a \times (b \times c)$

Commutative property of multiplication $a \times b = b \times a$

Distributive property of multiplication over addition $a \times (b + c) = a \times b + a \times c$

This web site has activities to help students more fully understand and retain new vocabulary (i.e. the definition page for *dice* actually generates rolls of the dice and gives students an opportunity to add them).

http://www.teachers.ash.org.au/jeather/maths/dictionary.html

Definitions and activities for these and other terms can be found on the Intermath website
http://intermath.coe.uga.edu/dictnary/homepg.asp

Polynomial Patterns

MCC9-12.A.APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. *(Focus on polynomial expressions that simplify to forms that are linear or quadratic in a positive integer power of x.)*

The following activity is a modification from NCTM's Illuminations Polynomial Puzzler http://illuminations.nctm.org/LessonDetail.aspx?id=L798

Can you find the pattern to the number puzzle below?

2	-6	-12
4	1	4
8	-6	-48

Explain the pattern.
Now, use the pattern to complete this table.

3	?	-15
	-2	?
		-240

HINT: Start with the question marks.

This can be expanded to multiplication with polynomials by solving the following:

1	x + 3	
-2x + 5	2	

What about this one?

-5		10x – 15
3x – 2		
	- 8x + 12	

Work the following on your own for 10 minutes, and then complete the tables with a partner.

1	x+ 7	
-2x + 5	2	

	$x-3$	
3	$-5x+1$	
		$30x^2-96x+18$

-4		-8
	$2x-6$	$-8x^2+72$

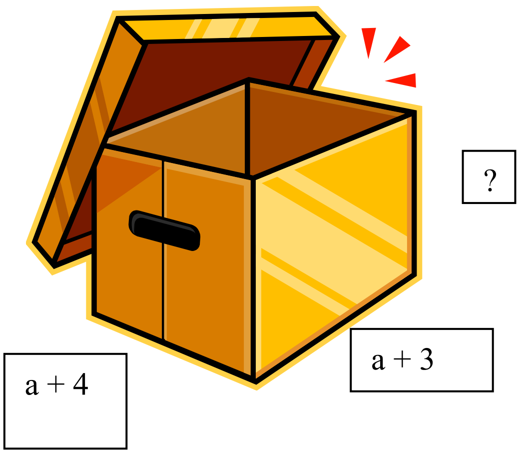
$x+3$		
2		$8x$
	$12x$	

		$2x+10$
$x+3$	7	
$2x+6$		

6		
	$x+3$	
18		$36x^2+144x+108$

Modeling

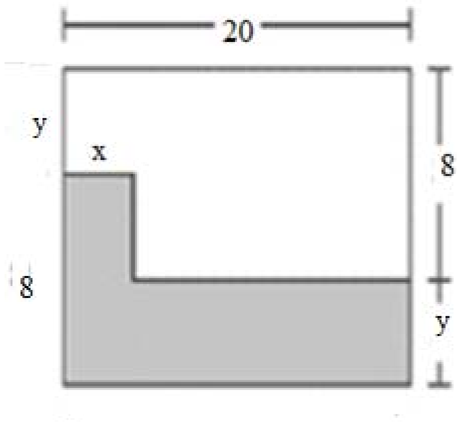
Problem A: The volume in cubic units of the box is $a^3 + 8a^2 + 19a + 12$. Its length is $a + 4$ units and its width is $a + 3$ units. What is its height?



Problem B

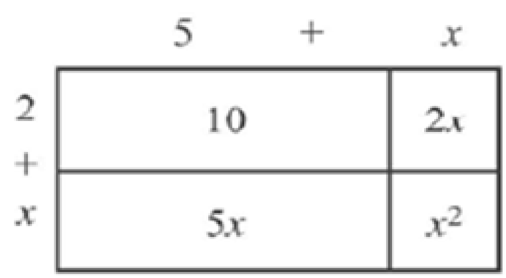
What is an illustration of $(x + 2)(x + 4)$?

Problem C: This rectangle shows the floor plan of an office. The shaded part of the plan is an area that is getting new tile. Write an algebraic expression that represents the area of the office that is getting new tile.



Problem D

What is the rectangle modeling?

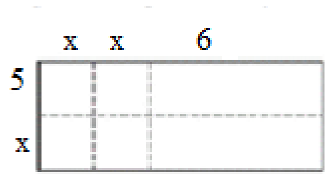


Problem E

Tyler and Susan each have a box that is the shape of a cube. The edges of Taylor's box are each x cm in length. The edges of Susan's box are 4 cm longer than on Tyler's cube. What binomial expression represents the volume of Susan's box?

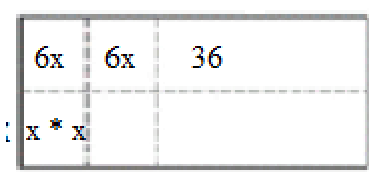
Problem F

What is the product of the expression represented by the model below?



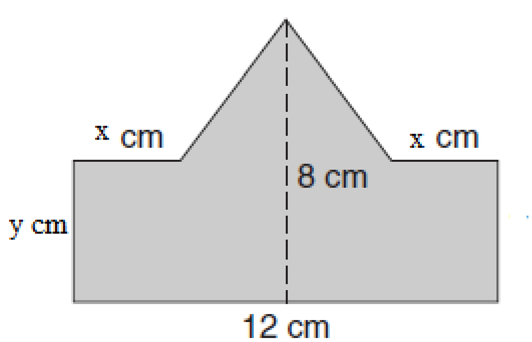
Problem G

Write the dimensions for the rectangle below.



Problem H

Find the area, including units, of the shape below.



How Long Does It Take?

MCC9-12.N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

MCC9-12.N.RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

MCC9-12.N.RN.3 Explain why the sum or product of rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Supplies Needed:

- Graphing Calculator
- Graph paper

Before sending astronauts to investigate the new planet of Exponentia, NASA decided to run a number of tests on the astronauts.

1. A specific multi-vitamin is eliminated from an adult male's bloodstream at a rate of about 20% per hour. The vitamin reaches peak level in the bloodstream of 300 milligrams.

   a. How much of the vitamin remains 2 hours after the peak level? 5 hours after the peak level? Make a table of values to record your answers. Write expressions for how you obtain your answers.

Time (hours) since peak	0	1	2	3	4	5
Vitamin concentration in bloodstream (mg)	300					

   b. Using your work from (a), write expressions for each computed value using the original numbers in the problem (300 and 20%). For example, after 2 hours, the amount of vitamin left is [300 * (1 - .2)]* (1 - .2).

   c. Using part (b), write a function for the vitamin level with respect to the number of hours after the peak level, *x*.

   d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a).

e. After how many hours will there be less than 10 mg of the vitamin remaining in the bloodstream? Explain how you would determine this answer using both a table feature and a graph.

f. Write an equation that you could solve to determine when the vitamin concentration is exactly 10 mg. Could you use the table feature to solve this equation? Could you use the graph feature? How could you use the intersection feature of your calculator? Solve the problem using one of these methods.

g. How would you solve the equation you wrote in (f) algebraically? What is the first step?

To finish solving the problem algebraically, we must know how to find inverses of exponential functions. This topic will be explored in more detail later.

2. A can of Instant Energy, a 16-ounce energy drink, contains 80 mg of caffeine. Suppose the caffeine in the bloodstream peaks at 80 mg. If ½ of the caffeine has been eliminated from the bloodstream after 5 hours (the half-life of caffeine in the bloodstream is 5 hours), complete the following:

   a. How much caffeine will remain in the bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Make a table to organize your answers. Explain how you came up with your answers. (Make a conjecture. You can return to your answers later to make any corrections.)

Time (hours) since peak	0	1	2	5	10
Caffeine in bloodstream (mg)	80				

   b. Unlike problem (1), in this problem in which 80% remained after each hour, in this problem, 50% remains after each **5 hours**.

      i. In problem (1), what did the exponent in your equation represent?

      ii. In this problem, our exponent needs to represent the number of 5-hour time periods that elapsed. If you represent 1 hour as 1/5 of a 5-hour time period, how do you represent 2 hours? 3 hours? 10 hours? *x* hours?

   c. Using your last answer in part (b) as your exponent, write an exponential function to model the amount of caffeine remaining in the blood stream *x* hours after the peak level.

d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a). (Be careful with your fractional exponents when entering in the calculator. Use parentheses.) If you need to, draw a line through your original answers in part (a) and list your new answers.

e. Determine the amount of caffeine remaining in the bloodstream 3 hours after the peak level? What about 8 hours after peak level? 20 hours? (Think about how many 5-hour intervals are in the number of hours you're interested in.)

f. Suppose the half-life of caffeine in the bloodstream was 3 hours instead of 5.

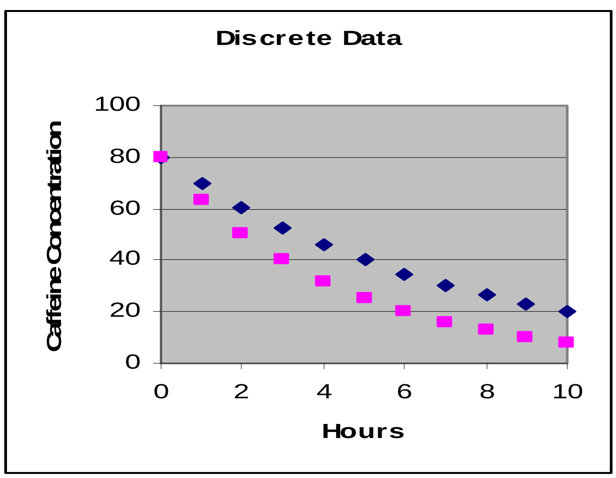
   i. Write a function for this new half-life time.

   ii. Determine the amount of caffeine in the bloodstream after 1 hour, 2 hours, 5 hours, and 10 hours. (You need to consider how many 3-hour time intervals are used in each time value.)

   iii. Which half-life time results in the caffeine being eliminated faster? Explain why this makes sense.

g. Graph both equations (from d and f) on graph paper. How are the graphs similar? Different? What are the intercepts? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?

Note that if we could only use integer exponents; e.g. 1, 2, 3, etc; our graphs would be discontinuous. We would have points (see right), rather than the smooth, continuous curve you graphed above.



It makes sense, in thinking about time, that we need all rational time values, e.g. 1/3 hour, 5/8 hour, etc. This raises the idea of rational exponents, that is, computing values such as $3^{3/4}$ or $(1/2)^{7/3}$.

3. **Rational Exponents.** In previous courses, you learned about different types of numbers and lots of rules of exponents.

   a. What are integers? Rational numbers? Which set of numbers is a subset of the other? Explain why this is true.

   b. Based on (a), what is the difference between integer exponents and rational exponents?

   c. Complete the following exponent rules. (If you don't remember the rules from your previous classes, try some examples to help you.)

For $a > 0$ and $b > 0$, and all values of m and n,

$a^0 =$ ___ $a^1 =$ ____ $a^n =$ ________________________

$(a^m)(a^n) =$ ________ $(a^m)/(a^n) =$ ________ $a^{-n} =$ _______

$(a^m)^n =$ ________ $(ab)^m =$ ___________ $(a/b)^m =$ __________

If $a^m = a^n$*, then* m ___ n.

The same rules you use for integer exponents also apply to rational exponents.

d. You have previously learned that the nth root of a number x can be represented as $x^{1/n}$.

   i. Using your rules of exponents, write another expression for $(x^{1/n})^m$.

   ii. Using your rules of exponents, write another expression for $(x^m)^{1/n}$.

   iii. What do you notice about the answers in (ii) and (iii)? What does this tell you about rational exponents?

This leads us to the definition of **rational exponents**.

For $a > 0$, and integers m and n, with $n > 0$,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m;\ a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}.$$

e. Rewrite the following using simplified rational exponents.

   i. $\sqrt[7]{x^3}$ ii. $\left(\dfrac{1}{x}\right)^{-5}$ iii. $\left(\sqrt{x}\right)^6$ iv. $\dfrac{1}{\sqrt[3]{x^5}}$

f. Simplify each of the following. Show your steps. For example, $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$.

i. $16^{3/4}$ ii. $36^{3/2}$ iii. $81^{7/4}$

Problems involving roots and rational exponents can sometimes be solved by rewriting expressions or by using inverses.

g. Consider the caffeine situation from above. The half-life of caffeine in an adult's bloodstream is 5 hours. How much caffeine is remains in the bloodstream each hour? Our original equation was $\boldsymbol{f(x) = 80(.5)^{x/5}}$**.** Use the rules of rational exponents to rewrite the equation so that the answer to the above question is given in the equation. What percent of the caffeine remains in the bloodstream each hour?

To solve equations such as $x^3 = 27$, we take the cube root of both sides. Alternately, we can raise both sides of the equation to the 1/3 power. That is, we raise both sides of the equation to the power that is the inverse (or reciprocal) of the power in the problem. To solve $x^{3/2} = 27$, we can either square both sides and then take the cube root, we can take the cube root of both sides and then square them, or we can raise both sides to the 2/3 power.

$x^{3/2} = 27 \rightarrow (x^{3/2})^{2/3} = 27^{2/3} \rightarrow x = (27^{1/3})^2 \rightarrow x = 3^2 = 9$

h. Rewrite each of the following using rational exponents and use inverses to solve for the variable. (You may need to use a calculator for some of them. Be careful!)

i. $\sqrt[5]{b} = 2$ ii. $\sqrt[5]{c^3} = 4.2$ iii. $\dfrac{1}{\sqrt[4]{d}} = \dfrac{1}{5}$

Let's look at some more problems that require the use of rational exponents.

4. Let's use a calculator to model bacteria growth. Begin with 25 bacteria.

a. If the number of bacteria doubles each hour, how many bacteria are alive after 1 hour? 2 hours?

b. Complete the chart below.

Time (hours)	0	1	2	3	4	5	6
Population	25	50					

c. Write a function that represents the population of bacteria after x hours. (Check that your function gives you the same answers you determined above. Think about what if means if the base number is 1. What type of base number is needed if the population is increasing?)

d. Use this expression to find the number of bacteria present after 7 ½ and 15 hours.

e. Suppose the initial population was 60 instead of 25. Write a function that represents the population of bacteria after x hours. Find the population after 7 1/2 hours and 15 hours.

f. Graph the functions in part (c) and (e). How are the graphs similar or different? What are the intercepts? What do the intercepts indicate? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?

g. Revisit the graphs in problem (2). Compare with the graphs above. How are they similar and different? How do the equations indicate if the graphs will be increasing or decreasing?

h. Consider the following: Begin with 25 bacteria. The number of bacteria doubles every 4 hours. Write a function, using a rational exponent, for the number of bacteria present after x hours.

i. Rewrite the function in (g), using the properties of exponents, so that the exponent is an integer. What is the rate of growth of the bacteria each hour?

j. If there are originally 25 bacteria, at what rate are they growing if the population of the bacteria doubles in 5 hours? (Hint: Solve $50 = 25(1 + r)^5$.) What about if the population triples in 5 hours? (Write equations and solve.)

k. If there are originally 25 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Solve the problem algebraically.

l. If there are originally 60 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Explain how you solved the problem. (Solving the problem algebraically will be addressed later in the unit.)

Imagine That!

MCC9-12.N.CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.

MCC9-12.N.CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

MCC9-12.N.CN.3 (+) Find the conjugate of a complex number; use conjugates to find ~~moduli and~~ quotients of complex numbers.

Complex Number System

The basic algebraic property of *i* is the following:

$i^2 = -1$

Let us begin with i^0, which is 1. Each power of *i* can be obtained from the previous power by multiplying it by *i*. We have:

$i^0 = 1$

$i^2 = -1$

$i^3 = i^2 * i = -1 * i = -i$

$i^4 = i^2 * i^2 = -1 * -1 = 1$

And we are back at 1 -- the cycle of powers will repeat! Any power of *i* will be either 1, *i*, −1, or −*i*

-- according to the remainder upon dividing the exponent *n* by 4.

i^n is

- 1 if the remainder is 0
- i if the remainder is 1
- -1 if the remainder is 2
- -i if the remainder is 3

To avoid having to memorize this algorithm, i to powers higher than two can be mathematically calculated using laws of exponents. They key is to create i squares.

$i^3 = i^2 * i = -1 * i = -i$

$i^4 = i^2 * i^2 = -1 * -1 = 1$

Examples Using Both Methods

$i^9 = i$, because when dividing 9 by 4, the remainder is 1. $i^9 = i^1$

OR $i^9 = i^2 * i^2 * i^2 * i^2 * i = -1 * -1 * -1 * -1 * i = i$

$i^{18} = -1$, because when dividing 18 by 4, the remainder is 2. $i^{18} = i^2$

OR, $i^{18} = (i^2)^9 = (-1)^9 = -1$. Using the exponent power rule, avoids having to write so many multiplications.

$i^{35} = -i$, because dividing 35 by 4, the remainder is 3. $i^{35} = i^3$

OR, $i^{35} = (i^{34})(i) = (i^2)^{17}(i) = (-1)^{17}(i) = -i$

$i^{40} = 1$, because on dividing 40 by 4, the remainder is 0. $i^{40} = i^0$.

OR, $i^{40} = (i^2)^{20} = (-1)^{20} = 1$

Note: Even powers of i will be either 1 or −1, since the exponent is a multiple of 4 or 2 more than a multiple of 4. Odd powers will be either i or $-i$.

Example 1. $3i \cdot 4i = 12i^2 = 12(-1) = -12$.

Example 2. $-5i \cdot 6i = -30i^2 = 30$.

We can see that the factor i^2 changes the sign of a product.

Evaluate the following.

1. $i * 2i$
2. $-5i * 4i$
3. $(3i)^2$

The complex number i is purely algebraic. That is, we call it a "number" because it will obey all the rules we normally associate with a number. We may add it, subtract it, multiply it, and and divide it. In fact, adding and subtracting are just like the basic algebra operations with variables. For example,

(6 + 3i) - (2 - 4i)= 6 + 3i - 2 + 4i = 4 + 7i. Notice that the negative sign or -1 was distributed through the second expression.

For multiplication, remember that the factor $i^2 = -1$

$(3 + 2i)(1 + 4i) = 3*1 + 3 *4i + 2i * 1 + 2i * 4i = 3 + 12i + 2i + 8i^2 = 3 + 14i + 8(-1) = -5 + 14i$.

$$\begin{aligned}(2 + 3i) \bullet (4 + 5i) &= 2(4 + 5i) + 3i(4 + 5i)\\ &= 8 + 10i + 12i + 15i^2\\ &= 8 + 22i + 15(-1)\\ &= 8 + 22i - 15\\ &= -7 + 22i\end{aligned}$$

Try: $(2 + 3i)(5 + 2i) = ?$

Did you get 4 + 19i?

The operation of division reviews all of the above operations

Complex conjugates

The complex conjugate of $a + bi$ is $a - bi$. The main point about a conjugate pair is that when they are multiplied --

$(a + bi)(a - bi)$

-- a positive real number is produced. That form is the difference of two squares:

$(a + bi)(a - bi) = a * a + a * (-bi) + bi * a + bi * (-bi) = a^2 - b^2i^2 = a^2 + b^2$

The product of a conjugate pair is equal to the <u>sum</u> of the squares of the components. Notice that the middle terms cancel. This fact helps us rationalize denominators, which is a form of division.

Example:

$$\frac{(4+2i)}{(3-i)} = \frac{(4+2i)}{(3-i)} \cdot \frac{(3+i)}{(3+i)}$$

$$= \frac{12+4i+6i+2i^2}{9+3i-3i-i^2} = \frac{12+10i+2(-1)}{9-(-1)}$$

$$= \frac{10+10i}{10} = \frac{1+i}{1} = 1+i$$

Please notice that the i's are removed from the denominators because they are square roots!

Please follow this link for kutasoftware worksheets on imaginary numbers:

http://www.kutasoftware.com/FreeWorksheets/Alg2Worksheets/Operations%20with%20Complex%20Numbers.pdf

For division:

http://www.kutasoftware.com/FreeWorksheets/Alg2Worksheets/Rationalizing%20Imaginary%20Denominators.pdf

If you really enjoy the imaginary number system, you can always have a t-shirt made with an *i* on it; in this way, you are truly using your imagination!

MORE REVIEW

Match the items.

a. 29/13 – 54i/13
b. -9 + 46i
c. – 12 + 10i
d. i
e. 7i
f. 18 – 6i
g. -10
h. 61 + 6i
i. 5/2

1. i^5, i^{17},i^{265}

2. 2i + 5i

3. 2i x 5i

4. 5i/2i

5. (3 + 2i) + (15 - 8i)

6. (3 + 2i) - (15 - 8i)

7. (3 + 2i)(15 - 8i)

8. (15 – 8i)/(3 + 2i)

9. $(3 + 2i)^3$